

Approximation by Rectangular Partial Sums of Double Conjugate Fourier Series¹

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We consider functions $f(x, y)$ bounded and measurable on the two-dimensional torus \mathbb{T}^2 . The conjugate function $\tilde{f}^{10}(x, y)$ with respect to the first variable is approximated by the rectangular partial sums $\tilde{s}_{mn}^{10}(f; x, y)$ of the corresponding conjugate series as m, n tend to ∞ independently of one another. Our goal is to estimate the rate of this approximation in terms of the oscillation of the function $\psi_{xy}^{10}(f; u, v) := f(x-u, y-v) - f(x+u, y-v) + f(x-u, y+v) - f(x+u, y+v)$ over appropriate subrectangles of \mathbb{T}^2 . In particular, we obtain a conjugate version of the well-known Dini–Lipschitz test on uniform convergence. We also give estimates in the case where the function $f(x, y)$ is of bounded variation in the sense of Hardy and Krause. Results of similar nature on the one-dimensional torus \mathbb{T} were proved in [7]. © 2000 Academic Press

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1. PRELIMINARIES: DOUBLE FOURIER SERIES

Given a function $f \in L^1(\mathbb{T}^2)$, its double Fourier series is defined by

$$\sum_{(j, k) \in \mathbb{Z}^2} c_{jk}(f) e^{i(jx + ky)}, \quad (1.1)$$

where $\mathbb{T} := [-\pi, \pi)$, $\mathbb{Z} := \{\dots, -1, 0, 1, 2, \dots\}$, and

$$c_{jk}(f) := \frac{1}{4\pi^2} \iint_{\mathbb{T}^2} f(u, v) e^{-i(ju + kv)} du dv. \quad (1.2)$$

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Denote by $s_{mn}(f; x, y)$ the (m, n) th symmetric rectangular partial sum of series (1.1). As it is well known, we have

$$s_{mn}(f; x, y) - f(x, y) = \frac{1}{\pi^2} \int_0^\pi \int_0^\pi \phi_{xy}(f; u, v) D_m(u) D_n(v) du dv,$$

where

$$\begin{aligned} \phi_{xy}(f; u, v) := & f(x - u, y - v) + f(x + u, y - v) \\ & + f(x - u, y + v) + f(x + u, y + v) - 4f(x, y) \end{aligned} \quad (1.3)$$

and

$$D_m(u) := \frac{1}{2} + \sum_{j=1}^m \cos ju = \frac{\sin(m + 1/2)u}{2 \sin u/2}, \quad m = 0, 1, 2, \dots,$$

is the Dirichlet kernel.

We shall also use the notations

$$\phi_x(f(\cdot, y); u) := f(x - u, y) + f(x + u, y) - 2f(x, y) \quad (1.4)$$

and

$$\phi_y(f(x, \cdot); v) := f(x, y - v) + f(x, y + v) - 2f(x, y). \quad (1.5)$$

For example, by (1.3) and (1.4), we have

$$\phi_x(f(\cdot, y); u) = \frac{1}{2} \phi_{xy}(f; u, 0).$$

We recall that the oscillation of a bounded function ϕ over an interval I (on the torus \mathbb{T}) is defined by

$$\text{osc}(\phi; I) := \sup \{ |\phi(u) - \phi(u')| : u, u' \in I \}.$$

In case ϕ is a bounded function of two variables, its oscillation over a rectangle $I_1 \times I_2$ (on the two-dimensional torus \mathbb{T}^2) is defined by

$$\begin{aligned} \text{osc}(\phi, I_1 \times I_2) := & \sup \{ |\phi(u, v) - \phi(u', v) - \phi(u, v') + \phi(u', v')| : \\ & u, u' \in I_1 \text{ and } v, v' \in I_2 \}. \end{aligned}$$

It will be always clear from the context whether an oscillation is formed over an interval (for a function of one variable) or over a rectangle (for a function of two variables).

In the sequel, we shall distinguish the following subintervals of $[0, \pi]$:

$$I_{jm} := \left[\frac{j\pi}{m+1}, \frac{(j+1)\pi}{m+1} \right], \quad j=0, 1, \dots, m; \quad m=0, 1, \dots$$

THEOREM 1.1 (Móricz [6]). *If $f \in L^1(\mathbb{T}^2)$ is bounded, then*

$$\begin{aligned} & |s_{mn}(f; x, y) - f(x, y)| \\ & \leq \left(1 + \frac{1}{\pi}\right)^2 \sum_{j=0}^m \sum_{k=0}^n \frac{1}{(j+1)(k+1)} \operatorname{osc}(\phi_{xy}(f); I_{jm} \times I_{kn}) \\ & \quad + \left(1 + \frac{1}{\pi}\right) \sum_{j=0}^m \frac{1}{j+1} \operatorname{osc}(\phi_x(f(\cdot, y)); I_{jm}) \\ & \quad + \left(1 + \frac{1}{\pi}\right) \sum_{k=0}^n \frac{1}{k+1} \operatorname{osc}(\phi_y(f(x, \cdot)); I_{kn}). \end{aligned} \quad (1.6)$$

We recall that the (total) modulus of continuity of a function f continuous on \mathbb{T}^2 is defined by

$$\begin{aligned} \omega(f; \delta_1, \delta_2) := \max\{ & |f(u, v) - f(u', v) - f(u, v') + f(u', v')| : \\ & |u - u'| \leq \delta_1 \text{ and } |v - v'| \leq \delta_2\}, \end{aligned}$$

while the partial moduli of f are defined by

$$\begin{aligned} \omega_x(f; \delta) & := \max\{|f(u, v) - f(u', v)| : |u - u'| \leq \delta \text{ and } v \in \mathbb{T}\}, \\ \omega_y(f; \delta) & := \max\{|f(u, v) - f(u, v')| : u \in \mathbb{T} \text{ and } |v - v'| \leq \delta\}, \end{aligned}$$

where $\delta, \delta_1, \delta_2 > 0$. Now, the extension of the Dini–Lipschitz test for double Fourier series follows immediately from Theorem 1.1.

COROLLARY 1.2. *If f is continuous on \mathbb{T}^2 ,*

$$\lim_{\delta_1, \delta_2 \rightarrow 0} \omega(f; \delta_1, \delta_2) \log \frac{1}{\delta_1} \log \frac{1}{\delta_2} = 0, \quad (1.7)$$

$$\lim_{\delta \rightarrow 0} \omega_x(f; \delta) \log \frac{1}{\delta} = 0, \quad (1.8)$$

$$\lim_{\delta \rightarrow 0} \omega_y(f; \delta) \log \frac{1}{\delta} = 0, \quad (1.9)$$

then $s_{mn}(f; x, y)$ converges uniformly to $f(x, y)$ as m and n tend to ∞ independently of one another.

In other words, we mean convergence of double series in Pringsheim's sense.

We recall that a function f of two variables is said to be of bounded variation over a rectangle $J_1 \times J_2$ in the sense of Hardy [3] and Krause (cf. the discussion in [4, Section 254]) if the following three conditions are satisfied.

(i) Given any two partitions

$$a = u_0 < u_1 < \cdots < u_m = b \quad \text{and} \quad c = v_0 < v_1 < \cdots < v_n = d$$

of the intervals $J_1 := [a, b]$ and $J_2 := [c, d]$, respectively, the total variation of f over $J_1 \times J_2$ defined by

$$\begin{aligned} \text{var}(f; J_1 \times J_2) := \sup \sum_{j=1}^m \sum_{k=1}^n & |f(u_{j-1}, v_{k-1}) - f(u_j, v_{k-1}) \\ & - f(u_{j-1}, v_k) + f(u_j, v_k)| \end{aligned}$$

is finite, where the supremum is extended over all partitions of J_1 and J_2 .

(ii) The restriction $f(\cdot, c)$ as a function of the first variable is of bounded variation over the interval J_1 .

(iii) The restriction $f(a, \cdot)$ as a function of the second variable is of bounded variation over J_2 .

Now, the extension of the Dirichlet–Jordan test for double Fourier series is a (nontrivial) consequence of Theorem 1.1.

COROLLARY 1.3 (Hardy [3]). *If f is of bounded variation on \mathbb{T}^2 in the sense of Hardy and Krause, then $s_{mn}(f; x, y)$ converges as $m, n \rightarrow \infty$ at each point (x, y) .*

2. MAIN RESULTS: DOUBLE CONJUGATE SERIES

One can associate three conjugate series to the double Fourier series (1.1):

$$\sum_{(j, k) \in \mathbb{Z}^2} (-i \operatorname{sign} j) c_{jk}(f) e^{i(jx + ky)} \quad (2.1)$$

(conjugate with respect to the first variable),

$$\sum_{(j, k) \in \mathbb{Z}^2} (-i \operatorname{sign} k) c_{jk}(f) e^{i(jx + ky)} \quad (2.2)$$

(conjugate with respect to the second variable), and

$$\sum_{(j,k) \in \mathbb{Z}^2} (-i \operatorname{sign} j)(-i \operatorname{sign} k) c_{jk}(f) e^{i(jx+ky)} \quad (2.3)$$

(conjugate with respect to both variables). We note that the formal addition of the double Fourier series (1.1), its conjugate series (2.1) multiplied by i , conjugate series (2.2) multiplied by i , and conjugate series (2.3) multiplied by $i^2 = -1$ results in a double power series on \mathbb{T}^2 (in terms of e^{ix} and e^{iy}):

$$c_{00}(f) + 2 \sum_{j=1}^{\infty} c_{j0}(f) e^{ijx} + 2 \sum_{k=1}^{\infty} c_{0k}(f) e^{iky} + 4 \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} c_{jk}(f) e^{i(jx+ky)}.$$

As it is well known, the corresponding conjugate functions are defined as follows:

$$\tilde{f}^{10}(x, y) := \lim_{h \rightarrow 0} \tilde{f}^{10}(h; x, y),$$

(conjugate with respect to the first variable), where

$$\tilde{f}^{10}(h; x, y) := \frac{1}{\pi} \int_h^\pi \frac{\psi_x(f(\cdot, y); u)}{2 \tan u/2} du, \quad h > 0, \quad (2.4)$$

and

$$\psi_x(f(\cdot, y); u) := f(x - u, y) - f(x + u, y); \quad (2.5)$$

furthermore,

$$\tilde{f}^{01}(x, y) := \lim_{h \rightarrow 0} \frac{1}{\pi} \int_h^\pi \frac{f(x, y - v) - f(x, y + v)}{2 \tan v/2} dv \quad (2.6)$$

(conjugate with respect to the second variable), and

$$\begin{aligned} \tilde{f}^{11}(x, y) := & \lim_{h_1, h_2 \rightarrow 0} \frac{1}{\pi^2} \int_{h_1}^\pi \int_{h_2}^\pi [f(x - u, y - v) - f(x + u, y - v) \\ & - f(x - u, y + v) + f(x + u, y + v)] \frac{du}{2 \tan u/2} \frac{dv}{2 \tan v/2} \quad (2.7) \end{aligned}$$

(conjugate with respect to both variables). That is, the integrals (2.4), (2.6), and (2.7) are taken in the sense of the ‘‘Cauchy principal value’’ at the points $x = 0$, or $y = 0$, or $x = y = 0$, respectively.

Privalov’s theorem (see, e.g. [10, Vol. II, p. 121]) immediately implies the a.e. existence of \tilde{f}^{10} and \tilde{f}^{01} under the assumption $f \in L^1(\mathbb{T}^2)$. The a.e. existence of \tilde{f}^{11} for $f \in L^1 \log^+ L(\mathbb{T}^2)$ was proved by Zygmund [9].

In the sequel, we shall not treat the conjugate series (2.2) separately. All the theorems concerning (2.1) can be reformulated with ease for (2.2), by taking their symmetric counterparts. As to the conjugate series (2.3), the case is different, and it will be treated in a subsequent paper.

We shall consider the symmetric rectangular partial sums of series (2.1) defined by

$$\tilde{s}_{mn}^{10}(f; x, y) := \sum_{|j| \leq m} \sum_{|k| \leq n} (-i \operatorname{sign} j) c_{jk}(f) e^{i(jx + ky)}.$$

It follows from (1.2) that

$$\tilde{s}_{mn}^{10}(f; x, y) = \frac{1}{\pi^2} \int_0^\pi \int_0^\pi \psi_{xy}^{10}(f; u, v) \tilde{D}_m(u) D_n(v) du dv, \tag{2.8}$$

where (cf. (1.3))

$$\begin{aligned} \psi_{xy}^{10}(f; u, v) := & f(x - u, y - v) - f(x + u, y - v) \\ & + f(x - u, y + v) - f(x + u, y + v), \end{aligned} \tag{2.9}$$

$$\tilde{D}_m(u) := \sum_{j=1}^m \sin j = \frac{1}{2 \tan u/2} - \frac{\cos(m + 1/2) u}{2 \sin u/2}, \quad m = 1, 2, \dots,$$

is the conjugate Dirichlet kernel, while $D_n(v)$ is the Dirichlet kernel. By (2.5) and (2.9), we have

$$\psi_x(f(\cdot, y); u) = \frac{1}{2} \psi_{xy}^{10}(f; u, 0). \tag{2.10}$$

Motivated by (1.5), we define

$$\begin{aligned} \phi_y \left(\tilde{f}^{10} \left(\frac{\pi}{m+1}; x, \cdot \right); v \right) := & \tilde{f}^{10} \left(\frac{\pi}{m+1}; x, y - v \right) + \tilde{f}^{10} \left(\frac{\pi}{m+1}; x, y + v \right) \\ & - 2\tilde{f}^{10} \left(\frac{\pi}{m+1}; x, y \right). \end{aligned} \tag{2.11}$$

Our main result is formulated in the following Theorem 2.1, which is the counterpart of Theorem 1.1 in the case of the conjugate series (2.1).

THEOREM 2.1. *If $f \in L^1(\mathbb{T}^2)$ is bounded, then*

$$\begin{aligned} & \left| \tilde{s}_{mn}^{10}(f; x, y) - \tilde{f}^{10}\left(\frac{\pi}{m+1}; x, y\right) \right| \\ & \leq \left(1 + \frac{1}{\pi}\right)^2 \sum_{j=0}^m \sum_{k=0}^n \frac{1}{(j+1)(k+1)} \operatorname{osc}(\psi_{xy}^{10}(f); I_{jm} \times I_{kn}) \\ & \quad + \left(1 + \frac{1}{\pi}\right) \sum_{j=0}^m \frac{1}{j+1} \operatorname{osc}(\psi_x(f(\cdot, y)); I_{jm}) \\ & \quad + \left(1 + \frac{1}{\pi}\right) \sum_{k=0}^n \frac{1}{k+1} \operatorname{osc}\left(\phi_y\left(\tilde{f}^{10}\left(\frac{\pi}{m+1}; x, \cdot\right)\right); I_{kn}\right). \end{aligned} \quad (2.12)$$

In comparison with (1.6) in Theorem 1.1, we observe the following:

(i) The first term on the right of (2.12) is almost the same as the first one in (1.6), with the exception that here $\psi_{xy}^{10}(f)$ stands instead of $\phi_{xy}(f)$.

(ii) The second term on the right of (2.12) is also almost identical with the second one in (1.6), but here $\psi_x(f(\cdot, y))$ stands instead of $\phi_x(f(\cdot, y))$.

(iii) The third term on the right of (2.12) is different from the third one in (1.6), being here $\phi_y(\tilde{f}^{10}(\pi/(m+1); x, \cdot))$ instead of $\phi_y(f(x, \cdot))$, and the former one depends also on m .

From Theorem 2.1 it follows immediately that if f is continuous on \mathbb{T}^2 , then

$$\begin{aligned} & \left| \tilde{s}_{mn}^{10}(f; x, y) - \tilde{f}^{10}\left(\frac{\pi}{m+1}; x, y\right) \right| \\ & \leq 4 \left(1 + \frac{1}{\pi}\right)^2 \omega\left(f; \frac{\pi}{m+1}, \frac{\pi}{n+1}\right) \log(em) \log(en) \\ & \quad + 2 \left(1 + \frac{1}{\pi}\right) \omega_x\left(f; \frac{\pi}{m+1}\right) \log(em) \\ & \quad + 2 \left(1 + \frac{1}{\pi}\right) \omega_y\left(\tilde{f}^{10}\left(\frac{\pi}{m+1}\right); \frac{\pi}{n+1}\right) \log(en), \end{aligned} \quad (2.13)$$

uniformly in x and y . In order to prove convergence of $\tilde{s}_{mn}^{10}(f; x, y)$, one has to ensure the existence of the conjugate function $\tilde{f}^{10}(x, y)$. This will be done in the proof of the next Corollary 2.2 (cf. (4.20)), which asserts the uniform convergence of the conjugate series (2.1).

COROLLARY 2.2. *If f is continuous on \mathbb{T}^2 ,*

$$\lim_{\delta \rightarrow 0} \omega_x(f; \delta) \left(\log \frac{1}{\delta} \right)^2 = 0, \quad (2.14)$$

$$\lim_{\delta \rightarrow 0} \omega_y(f; \delta) \left(\log \frac{1}{\delta} \right)^2 = 0, \quad (2.15)$$

then $\tilde{s}_{mn}^{10}(f; x, y)$ converges uniformly to $\tilde{f}^{10}(x, y)$ as $m, n \rightarrow \infty$. In particular, the conjugate function \tilde{f}^{10} exists everywhere and is continuous on \mathbb{T}^2 .

It is plain that the conditions in Corollary 2.2 are stronger than those in Corollary 1.2. It is instructive to see that the conclusion of Corollary 2.2 can be proved under essentially weaker conditions in the particular case where

$$f(x, y) := g_1(x) g_2(y). \quad (2.16)$$

COROLLARY 2.2*. *If g_1 and g_2 in (2.16) are continuous on \mathbb{T} ,*

$$\int_0^1 \frac{\omega(g_1; \delta)}{\delta} d\delta < \infty, \quad (2.17)$$

$$\lim_{\delta \rightarrow 0} \omega(g_2; \delta) \log \frac{1}{\delta} = 0, \quad (2.18)$$

then $\tilde{s}_{mn}^{10}(f; x, y)$ converges uniformly to $\tilde{f}^{10}(x, y) = \tilde{g}_1(x) g_2(y)$ as $m, n \rightarrow \infty$.

Condition (2.17) is satisfied if, for example,

$$\omega(g_1; \delta) = \mathcal{O} \left\{ \log \frac{1}{\delta} \left(\log \log \frac{1}{\delta} \right)^{1+\varepsilon} \right\}^{-1} \quad \text{as } \delta \rightarrow 0, \quad (2.19)$$

where $\varepsilon > 0$ is fixed. We note that condition (2.17) is the best possible to ensure the existence of the conjugate function $\tilde{g}_1(x)$ at each point x . Namely, if $\omega(\delta)$ is a concave modulus of continuity such that

$$\int_0^1 \frac{\omega(\delta)}{\delta} d\delta = \infty,$$

then there exists a continuous function g on \mathbb{T} such that

$$\omega(g; \delta) = \mathcal{O}\{\omega(\delta)\} \quad \text{as } \delta \rightarrow 0$$

and the conjugate function $\tilde{g}(x)$ does not exist at $x=0$. For details, see [7, pp. 210–211].

We shall also deduce from Theorem 2.1 the next Corollary 2.3, which is the extension of a result by Mazhar and Al-Budaiwi [5] from single to double conjugate series.

COROLLARY 2.3. *If f is of bounded variation on \mathbb{T}^2 in the sense of Hardy and Krause, then*

$$\begin{aligned} & \left| \tilde{s}_{mn}^{10}(f; x, y) - \tilde{f}^{10}\left(\frac{\pi}{m+1}; x, y\right) \right| \\ & \leq 16 \left(1 + \frac{1}{\pi}\right)^2 \frac{1}{(m+1)(n+1)} \\ & \quad \times \sum_{j=0}^m \sum_{k=0}^n \text{var}\left(\psi_{xy}^{10}(f); \left[0, \frac{\pi}{j+1}\right] \times \left[0, \frac{\pi}{k+1}\right]\right) \\ & \quad + 4 \left(1 + \frac{1}{\pi}\right) \frac{1}{m+1} \sum_{j=0}^m \text{var}\left(\psi_x(f(\cdot, y)); \left[0, \frac{\pi}{j+1}\right]\right) \\ & \quad + 4 \left(1 + \frac{1}{\pi}\right) \frac{1}{n+1} \sum_{k=0}^n \text{var}\left(\phi_y\left(\tilde{f}^{10}\left(\frac{\pi}{m+1}; x, \cdot\right)\right); \left[0, \frac{\pi}{k+1}\right]\right). \end{aligned} \tag{2.20}$$

We note that if f is of bounded variation on \mathbb{T}^2 , then for each $h > 0$, $\tilde{f}^{10}(h; x, \cdot)$ is also of bounded variation on \mathbb{T} . Indeed, a rough estimate gives

$$\text{var}(\tilde{f}^{10}(h; x, \cdot), \mathbb{T}) \leq \{\text{var}(f; \mathbb{T}^2) + \text{var}(f(x, \cdot); \mathbb{T})\} \log \frac{\pi}{h}.$$

In the general case, we are unable to draw a reasonable convergence result from Corollary 2.3. However, in the particular case of (2.16), we can deduce the following extension of Young's test [8] (see also [10, Vol. I, p. 59]).

COROLLARY 2.4. *If g_1 and g_2 in (2.16) are of bounded variation on \mathbb{T} , g_2 is regularized at the point y in the sense that*

$$g_2(y) = \frac{1}{2}\{g_2(y-0) + g_2(y+0)\},$$

and $g_2(y) \neq 0$, then $\tilde{s}_{mn}^{10}(f)$ converges as $m, n \rightarrow \infty$ at the point (x, y) if and only if $\tilde{f}^{10}(x, y)$ exists, in which case $\tilde{f}^{10}(x, y) = \tilde{g}_1(x) g_2(y)$ exists.

3. AUXILIARY RESULTS

Given a function $g \in L^1(\mathbb{T})$, its Fourier series is defined by

$$\sum_{j \in \mathbb{Z}} d_j(g) e^{ijx}, \quad d_j(g) := \frac{1}{2\pi} \int_{\mathbb{T}} g(u) e^{-iju} du. \quad (3.1)$$

Denote by $s_m(g; x)$ the m th symmetric partial sum of series (3.1). As it is well known, we have

$$s_m(g; x) - g(x) = \frac{1}{\pi} \int_0^\pi \phi_x(g; u) D_m(u) du,$$

where

$$\phi_x(g; u) := g(x-u) + g(x+u) - 2g(x). \quad (3.2)$$

LEMMA 3.1 (Bojanic and Waterman [2]). *If $g \in L^1(\mathbb{T})$ is bounded, then*

$$|s_m(g; x) - g(x)| \leq \left(1 + \frac{1}{\pi}\right) \sum_{j=0}^m \frac{1}{j+1} \text{osc}(\phi_x(g); I_{jm}).$$

Actually, the factor $1 + 2/\pi$ occurs in [2] instead of $1 + 1/\pi$.

We recall that the conjugate series to (3.1) is defined by

$$\sum_{j \in \mathbb{Z}} (-i \text{sign } j) c_j(g) e^{ijx}. \quad (3.3)$$

Denote by $\tilde{s}_m(g; x)$ the m th symmetric partial sum of series (3.3). As it is well known, we have

$$\tilde{s}_m(g; x) = \frac{1}{\pi} \int_0^\pi \psi_x(g; u) \tilde{D}_m(u) du,$$

where

$$\psi_x(g; u) := g(x-u) - g(x+u). \quad (3.4)$$

Motivated by this, the conjugate function to g is defined as a Cauchy principal value integral:

$$\tilde{g}(x) := \lim_{h \rightarrow 0} \tilde{g}(h; x),$$

where

$$\tilde{g}(h; x) := \frac{1}{\pi} \int_h^\pi \frac{\psi_x(g; u)}{2 \tan u/2} du, \quad h > 0.$$

Privalov's theorem states that \tilde{g} exists almost everywhere whenever $g \in L^1(\mathbb{T})$.

LEMMA 3.2 (Móricz [7]). *If $f \in L^1(\mathbb{T})$ is bounded, then*

$$\left| \tilde{s}_m(g; x) - \tilde{g}\left(\frac{\pi}{m+1}; x\right) \right| \leq \left(1 + \frac{1}{\pi}\right) \sum_{j=0}^m \frac{1}{j+1} \text{osc}(\phi_x(g); I_{jm}).$$

LEMMA 3.3. *If $\phi \in L^1(\mathbb{T}^2)$ is bounded then*

$$\begin{aligned} & \left| \frac{1}{\pi^2} \int_{\pi/(m+1)}^\pi \int_0^\pi [\phi(u, v) - \phi(0, v) - \phi(u, 0) + \phi(0, 0)] \right. \\ & \quad \times \left. \frac{\cos(m+1/2)u}{2 \sin u/2} D_n(v) du dv \right| \\ & \leq \left(1 + \frac{1}{\pi}\right)^2 \sum_{j=1}^m \sum_{k=0}^n \frac{1}{(j+1)(k+1)} \text{osc}(\phi; I_{jm} \times I_{kn}) \\ & \quad + \left(\frac{1}{\pi} + \frac{1}{\pi^2}\right) \sum_{k=0}^n \frac{1}{k+1} \text{osc}(\phi, I_{0m} \times I_{kn}). \end{aligned} \quad (3.5)$$

Proof. First, analysing the proof of [6, Theorem 2, especially that of (5.18)] reveals that (3.5) holds true if $D_m(u) = (\sin(m+1/2)u)/(2 \sin u/2)$ stands instead of $(\cos(m+1/2)u)/(2 \sin u/2)$ on the left-hand side.

Second, we observe that only the inequality

$$|D_m(u)| \leq \frac{\pi}{2u}, \quad 0 < u \leq \pi,$$

is used in the proof of [6, Theorem 2] in the cases where $u \in I_{jm}$ for some $j \geq 1$. But the same inequality holds true for $(\cos(m+1/2)u)/(2 \sin u/2)$, as well. ■

A simple application of the second mean-value theorem gives the following

LEMMA 3.4. *For all $0 < y \leq \pi$ and $n \geq 1$,*

$$\left| \int_y^\pi D_n(v) dv \right| \leq \frac{\pi}{(n+1/2)y}.$$

For example, a proof of Lemma 3.4 can be found in [2], where $n + 1$ stands instead of $n + 1/2$.

LEMMA 3.5. *If ϕ is of bounded variation over $[0, \pi]$, then*

$$\sum_{j=0}^m \frac{1}{j+1} \operatorname{osc}(\phi, I_{jm}) \leq \frac{2}{m+1} \sum_{j=0}^{m-1} \operatorname{var} \left(\phi; \left[0, \frac{\pi}{j+1} \right] \right). \quad (3.6)$$

This lemma with another constant was first proved in [2] in a more general setting, namely for a function ϕ of generalized bounded variation. Now, we present a shorter proof, without relying on Stieltjes integral.

Proof. The case $m := 0$ is trivial, since

$$\operatorname{osc}(\phi; [0, \pi]) \leq \operatorname{var}(\phi; [0, \pi]).$$

Assume $m \geq 1$. Clearly, for $j \geq 0$, we have

$$\operatorname{osc}(\phi; I_{jm}) \leq \operatorname{var} \left(\phi; \left[0, \frac{(j+1)\pi}{m+1} \right] \right) - \operatorname{var} \left(\phi; \left[0, \frac{j\pi}{m+1} \right] \right)$$

(with the agreement that $\operatorname{var}(\phi; [0, 0]) = 0$ in case $j = 0$). Introduce the notation

$$a_j := \operatorname{var} \left(\phi; \left[0, \frac{j\pi}{m+1} \right] \right), \quad j = 1, 2, \dots, m+1, \quad (3.7)$$

then, by summation by parts, we get

$$\begin{aligned} \sum_{j=0}^m \frac{1}{j+1} \operatorname{osc}(\phi; I_{jm}) &\leq a_1 + \sum_{j=1}^m \frac{1}{j+1} (a_{j+1} - a_j) \\ &= \sum_{j=1}^m \left(\frac{1}{j} - \frac{1}{j+1} \right) a_j + \frac{a_{m+1}}{m+1}. \end{aligned} \quad (3.8)$$

Since the function $\operatorname{var}(\phi; [0, t])$ is nondecreasing in $t > 0$, two simple estimates and an integration by substitution yield

$$\begin{aligned} \sum_{j=1}^m \left(\frac{1}{j} - \frac{1}{j+1} \right) a_j &\leq \int_1^{m+1} \operatorname{var} \left(\phi; \left[0, \frac{t\pi}{m+1} \right] \right) \frac{dt}{t^2} \\ &= \frac{1}{m+1} \int_1^{m+1} \operatorname{var} \left(\phi; \left[0, \frac{\pi}{s} \right] \right) ds \\ &\leq \frac{1}{m+1} \sum_{j=0}^{m-1} \operatorname{var} \left(\phi; \left[0, \frac{\pi}{j+1} \right] \right). \end{aligned} \quad (3.9)$$

Putting (3.8) and (3.9) together (while keeping notation (3.7) in mind) gives immediately inequality (3.6). ■

LEMMA 3.6. *If ϕ is of bounded variation over $[0, \pi] \times [0, \pi]$, then*

$$\begin{aligned} & \sum_{j=0}^m \sum_{k=0}^n \frac{1}{(j+1)(k+1)} \operatorname{osc}(\phi; I_{jm} \times I_{kn}) \\ & \leq \frac{4}{(m+1)(n+1)} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \operatorname{var} \left(\phi; \left[0, \frac{\pi}{j+1} \right] \times \left[0, \frac{\pi}{k+1} \right] \right). \end{aligned} \quad (3.10)$$

Proof. In case $m=0$ or $n=0$, (3.10) follows from Lemma 3.5. Assume $m \geq 1$ and $n \geq 1$. Clearly, for $j \geq 1$ and $k \geq 1$, we have

$$\operatorname{osc}(\phi; I_{jm} \times I_{kn}) \leq a_{j+1, k+1} - a_{j, k+1} - a_{j+1, k} + a_{jk},$$

where

$$a_{jk} := \operatorname{var} \left(\phi; \left[0, \frac{j\pi}{m+1} \right] \times \left[0, \frac{k\pi}{n+1} \right] \right); \quad (3.11)$$

while for $j \geq 1$ and $k=0$, we have

$$\operatorname{osc}(\phi; I_{jm} \times I_{0n}) \leq a_{j+1, 1} - a_{j1};$$

and its symmetric counterpart when $j=0$ and $k \geq 1$. Then, by a double summation by parts, we obtain

$$\begin{aligned} & \sum_{j=0}^m \sum_{k=0}^n \frac{1}{(j+1)(k+1)} \operatorname{osc}(\phi; I_{jm} \times I_{kn}) \\ & \leq a_{11} + \sum_{j=1}^m \frac{1}{j+1} (a_{j+1, 1} - a_{j1}) + \sum_{k=1}^n \frac{1}{k+1} (a_{1, k+1} - a_{1k}) \\ & \quad + \sum_{j=1}^m \sum_{k=1}^n \frac{1}{(j+1)(k+1)} (a_{j+1, k+1} - a_{j, k+1} - a_{j+1, k} + a_{jk}) \\ & = \sum_{j=1}^m \sum_{k=1}^n \left(\frac{1}{j} - \frac{1}{j+1} \right) \left(\frac{1}{k} - \frac{1}{k+1} \right) a_{jk} \\ & \quad + \frac{1}{n+1} \sum_{j=1}^m \left(\frac{1}{j} - \frac{1}{j+1} \right) a_{j, n+1} \\ & \quad + \frac{1}{m+1} \sum_{k=1}^n \left(\frac{1}{k} - \frac{1}{k+1} \right) a_{m+1, k} + \frac{a_{m+1, n+1}}{(m+1)(n+1)}. \end{aligned} \quad (3.12)$$

Since the function $\text{var}(\phi; [0, u] \times [0, v])$ is nondecreasing in both u and v , an analogous argument which led to (3.9) yields

$$\begin{aligned} & \sum_{j=1}^m \sum_{k=1}^n \left(\frac{1}{j} - \frac{1}{j+1} \right) \left(\frac{1}{k} - \frac{1}{k+1} \right) a_{jk} \\ & \leq \int_1^{m+1} \int_1^{n+1} \text{var} \left(\phi; \left[0, \frac{u\pi}{m+1} \right] \times \left[0, \frac{v\pi}{n+1} \right] \right) \frac{du}{u^2} \frac{dv}{v^2} \\ & = \frac{1}{(m+1)(n+1)} \int_1^{m+1} \int_1^{n+1} \text{var} \left(\phi; \left[0, \frac{\pi}{s} \right] \times \left[0, \frac{\pi}{t} \right] \right) ds dt \\ & \leq \frac{1}{(m+1)(n+1)} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \text{var} \left(\phi; \left[0, \frac{\pi}{j+1} \right] \times \left[0, \frac{\pi}{k+1} \right] \right). \quad (3.13) \end{aligned}$$

According to (3.6), we have

$$\sum_{j=1}^m \left(\frac{1}{j} - \frac{1}{j+1} \right) a_{j, n+1} \leq \frac{2}{m+1} \sum_{j=0}^{m-1} \text{var} \left(\phi; \left[\frac{\pi}{j+1} \right] \times [0, \pi] \right)$$

and

$$\sum_{k=1}^n \left(\frac{1}{k} - \frac{1}{k+1} \right) a_{m+1, k} \leq \frac{2}{n+1} \sum_{k=0}^{m-1} \text{var} \left(\phi; [0, \pi] \times \left[0, \frac{\pi}{k+1} \right] \right).$$

Finally, it is plain that

$$a_{m+1, n+1} \leq \text{var}(\phi; [0, \pi] \times [0, \pi]).$$

Combining the last three inequalities with (3.12) and (3.13) (and keeping notation (3.11) in mind) gives (3.10). ■

4. PROOFS

Proof of Theorem 2.1. We start with representation (2.8) and write $\psi = \psi(u, v)$ instead of $\psi_{xy}^{10}(f; u, v)$. Since

$$\frac{1}{2}\psi(u, 0) = f(x-u, y) - f(x+u, y)$$

(cf. (2.5) and (2.10)), we have

$$\begin{aligned} \tilde{s}_{mm}^{10}(f; x, y) &= \frac{1}{\pi^2} \int_0^\pi \int_0^\pi [\psi(u, v) - \psi(u, 0)] \tilde{D}_m(u) D_n(v) du dv \\ &\quad + \frac{1}{\pi} \int_0^\pi \frac{1}{2} \psi(u, 0) \tilde{D}_m(u) du, \end{aligned}$$

whence

$$\begin{aligned} \tilde{s}_{mm}^{10}(f; x, y) - \tilde{f}^{10}\left(\frac{\pi}{m+1}; x, y\right) &= \frac{1}{\pi^2} \int_0^\pi \int_0^\pi [\psi(u, v) - \psi(u, 0)] \tilde{D}_m(u) D_n(v) du dv \\ &\quad + \left[\tilde{s}_m(f(\cdot, y), x) - \tilde{f}^{10}\left(\frac{\pi}{m+1}; x, y\right) \right] =: A_{mm} + B_m, \end{aligned} \quad (4.1)$$

say. Applying Lemma 3.2 (cf. (2.5) and (3.4)) gives

$$|B_m| \leq \left(1 + \frac{1}{\pi}\right) \sum_{j=0}^m \frac{1}{j+1} \text{osc}(\psi_x(f(\cdot, y)); I_{jm}). \quad (4.2)$$

In the sequel, we shall estimate

$$A_{mm} := \frac{1}{\pi^2} \int_0^\pi \int_0^\pi g(u, v) \tilde{D}_m(u) D_n(v) du dv, \quad (4.3)$$

where the auxiliary function g is defined by

$$g(u, v) := \psi(u, v) - \phi(u, 0). \quad (4.4)$$

Clearly, $g(u, 0) = g(0, v) = 0$ for all u, v . Besides, we shall use the notation

$$\theta_{kn} := \frac{k\pi}{n+1}, \quad k = 0, 1, \dots, n+1; \quad n = 1, 2, \dots \quad (4.5)$$

It is plain that $I_{kn} = [\theta_{kn}, \theta_{k+1, n}]$.

We decompose the double integral in (4.3) as follows:

$$\begin{aligned}
 A_{mn} &= \frac{1}{\pi^2} \int_0^{\pi/(m+1)} \int_0^\pi g(u, v) \tilde{D}_m(u) D_n(v) du dv \\
 &\quad + \frac{1}{\pi^2} \int_{\pi/(m+1)}^\pi \int_0^\pi \frac{g(u, v)}{2 \tan u/2} D_n(v) dv \\
 &\quad - \frac{1}{\pi^2} \int_{\pi/(m+1)}^\pi \int_0^\pi g(u, v) \frac{\cos(m+1/2)u}{2 \sin u/2} D_n(v) du dv \\
 &=: A_{mn}^1 + A_{mn}^2 + A_{mn}^3, \quad \text{say.}
 \end{aligned} \tag{4.6}$$

By Lemma 3.3, we have

$$\begin{aligned}
 |A_{mn}^3| &\leq \left(1 + \frac{1}{\pi}\right)^2 \sum_{j=1}^m \sum_{k=0}^n \frac{1}{(j+1)(k+1)} \text{osc}(\psi; I_{jm} \times I_{kn}) \\
 &\quad + \left(\frac{1}{\pi} + \frac{1}{\pi^2}\right) \sum_{k=0}^n \frac{1}{k+1} \text{osc}(\psi; I_{0m} \times I_{kn}).
 \end{aligned} \tag{4.7}$$

By (2.4), (2.5), and (4.4), we may write that

$$\begin{aligned}
 A_{mn}^2 &= \frac{1}{\pi} \int_0^\pi \left\{ \frac{1}{\pi} \int_{\pi/(m+1)}^\pi \frac{g(u, v)}{2 \tan u/2} du \right\} D_n(v) dv \\
 &= \frac{1}{\pi} \int_0^\pi \left\{ \tilde{f}^{10} \left(\frac{\pi}{m+1}; x, y-v \right) + \tilde{f}^{10} \left(\frac{\pi}{m+1}; x, y+v \right) \right. \\
 &\quad \left. - 2\tilde{f}^{10} \left(\frac{\pi}{m+1}; x, y \right) \right\} D_n(v) dv.
 \end{aligned}$$

We observe that the integral on the right is the difference between the truncated conjugate function $\tilde{f}^{10}(\pi/(m+1); x, \cdot)$ and the n th partial sum of its Fourier series with respect to the second variable y , while the first variable x is fixed:

$$A_{mn}^2 = s_n \left(\tilde{f}^{10} \left(\frac{\pi}{m+1}; x, \cdot \right); y \right) - \tilde{f}^{10} \left(\frac{\pi}{m+1}; x, y \right).$$

By Lemma 3.1 (cf. (2.11) and (3.2)), we find

$$|A_{mn}^2| \leq \left(1 + \frac{1}{\pi}\right) \sum_{k=0}^n \frac{1}{k+1} \text{osc} \left(\phi_y \left(\tilde{f}^{10} \left(\frac{\pi}{m+1}; x, \cdot \right) \right); I_{kn} \right), \tag{4.8}$$

where $\phi_y(\tilde{f}^{10}(\pi/(m+1); x, \cdot))$ is defined in (2.11).

It only remains to estimate A_{mn}^1 . We decompose it as follows:

$$\begin{aligned}
 A_{mn}^1 &= \frac{1}{\pi^2} \sum_{k=0}^n \int_{I_{0m}} \int_{I_{kn}} g(u, v) \tilde{D}_m(u) D_n(v) du dv \\
 &= \frac{1}{\pi^2} \int_{I_{0m}} \int_{I_{0n}} g(u, v) \tilde{D}_m(u) D_n(v) du dv \\
 &\quad + \frac{1}{\pi^2} \sum_{k=1}^n \int_{I_{0m}} \int_{I_{kn}} [g(u, v) - g(u, \theta_{kn})] \tilde{D}_m(u) D_n(v) du dv \\
 &\quad + \frac{1}{\pi^2} \sum_{k=1}^n \int_{I_{0m}} \int_{I_{kn}} g(u, \theta_{kn}) \tilde{D}_m(u) D_n(v) du dv \\
 &=: A_{mn}^{11} + A_{mn}^{12} + A_{mn}^{13}, \tag{4.9}
 \end{aligned}$$

say, where θ_{kn} is defined in (4.5). In the sequel, we shall frequently use the elementary inequalities:

$$|\tilde{D}_m(u)| \leq \min \left\{ m, \frac{\pi}{u} \right\}$$

and

$$|D_m(u)| \leq \min \left\{ m + \frac{1}{2}, \frac{\pi}{2u} \right\} \quad \text{for } 0 < u \leq \pi. \tag{4.10}$$

By these and (4.4), we have

$$|A_{mn}^{11}| \leq \text{osc}(\psi; I_{0m} \times I_{0n}). \tag{4.11}$$

By (4.4) and the oddness of $\psi(u, v)$ in u , we may write that

$$g(u, v) - g(u, \theta_{kn}) = \psi(u, v) - \psi(u, \theta_{kn}) - \psi(0, v) + \psi(0, \theta_{kn}). \tag{4.12}$$

By this and (4.10), we have

$$\begin{aligned}
 |A_{mn}^{12}| &\leq \frac{1}{\pi} \sum_{k=1}^n \text{osc}(\psi; I_{0m} \times I_{kn}) \int_{I_{kn}} \frac{\pi}{2v} dv \\
 &\leq \sum_{k=1}^n \frac{1}{k+1} \text{osc}(\psi; I_{0m} \times I_{kn}). \tag{4.13}
 \end{aligned}$$

Setting

$$R_{kn} := \int_{\theta_{kn}}^{\pi} D_n(v) dv,$$

by virtue of Lemma 3.4, we see that

$$|R_{kn}| \leq \frac{1}{k} \quad \text{for } k = 1, 2, \dots, n, \quad \text{and } R_{n+1, n} = 0. \quad (4.14)$$

Performing a summation by parts gives

$$\begin{aligned} \sum_{k=1}^n \int_{I_{kn}} g(u, \theta_{kn}) D_n(v) dv &= \sum_{k=1}^n g(u, \theta_{kn})(R_{kn} - R_{k+1, n}) \\ &= \sum_{k=1}^n R_{kn} [g(u, \theta_{kn}) - g(u, \theta_{k-1, n})], \end{aligned}$$

where u is fixed. Writing out the difference $g(u, \theta_{kn}) - g(u, \theta_{k-1, n})$ in the same manner as in (4.12), while using (4.14), hence we conclude that

$$\begin{aligned} |A_{mn}^{13}| &= \left| \frac{1}{\pi^2} \int_{I_{0m}} \left\{ \sum_{k=1}^n \int_{I_{kn}} g(u, \theta_{kn}) D_n(v) dv \right\} \tilde{D}_m(u) du \right| \\ &\leq \frac{1}{\pi} \sum_{k=1}^n \frac{1}{k} \text{osc}(\psi; I_{0m} \times I_{k-1, n}) \\ &= \frac{1}{\pi} \sum_{k=0}^{n-1} \frac{1}{k+1} \text{osc}(\psi; I_{0m} \times I_{kn}). \end{aligned} \quad (4.15)$$

Combining (4.1), (4.2), (4.6)–(4.9), (4.11), (4.13), and (4.15) yields (2.12). ■

Proof of Corollary 2.2. First, we prove that if $f \in L^1(\mathbb{T}^2)$ is bounded and

$$\sum_{j=0}^{\infty} \frac{1}{j+1} \text{osc} \left(\psi_x(f(\cdot, y)); \left[0, \frac{\pi}{j+1} \right] \right) < \infty, \quad (4.16)$$

then the conjugate function \tilde{f}^{10} exists at the point (x, y) . In fact, assume

$$\frac{\pi}{m+1} < h \leq \frac{\pi}{m} \quad \text{for some } m \geq 1, \quad (4.17)$$

then

$$\begin{aligned} \left| \tilde{f}^{10}(h; x, y) - \tilde{f}^{10}\left(\frac{\pi}{m+1}; x, y\right) \right| &\leq \frac{1}{\pi} \int_{\pi/(m+1)}^{\pi/m} \frac{|\psi_x(f(\cdot, y)); u|}{2 \tan u/2} du \\ &\leq \frac{1}{m\pi} \operatorname{osc} \left(\psi_x(f(\cdot, y)); \left[0, \frac{\pi}{m}\right] \right). \end{aligned} \quad (4.18)$$

Analogously, we have

$$\begin{aligned} \left| \tilde{f}^{10}\left(\frac{\pi}{m+1}; x, y\right) - \tilde{f}^{10}(x, y) \right| &\leq \sum_{j=m+1}^{\infty} \left| \tilde{f}^{10}\left(\frac{\pi}{j}; x, y\right) - \tilde{f}^{10}\left(\frac{\pi}{j+1}; x, y\right) \right| \\ &\leq \sum_{j=m+1}^{\infty} \frac{1}{j\pi} \operatorname{osc} \left(\psi_x(f(\cdot, y)); \left[0, \frac{\pi}{j}\right] \right). \end{aligned} \quad (4.19)$$

Now, from (4.16)–(4.19) it follows that $\tilde{f}^{10}(x, y)$ exists and

$$|\tilde{f}^{10}(h; x, y) - \tilde{f}^{10}(x, y)| \leq \frac{1}{\pi} \sum_{j=m}^{\infty} \frac{1}{j} \operatorname{osc} \left(\psi_x(f(\cdot, y)); \left[0, \frac{\pi}{j}\right] \right). \quad (4.20)$$

It is plain that (4.16) is satisfied uniformly in (x, y) if f is continuous on \mathbb{T}^2 and

$$\sum_{j=0}^{\infty} \frac{1}{j+1} \omega_x \left(f; \frac{\pi}{j+1} \right) < \infty,$$

or equivalently,

$$\int_0^{\pi} \frac{\omega_x(f; \delta)}{\delta} d\delta < \infty. \quad (4.21)$$

Clearly, (4.21) follows from (2.14). (However, the order of magnitude occurring on the right-hand side of (2.19) would be enough here.)

We note that condition (4.21) is also the best possible in order to ensure the existence of $\tilde{f}^{10}(x, y)$ at each point (x, y) , in the sense of the remark made after Corollary 2.2*.

In order to complete the proof, it is enough to consider inequality (2.13), which is a direct consequence of Theorem 2.1. Due to (2.14) and (2.15), the first two terms on the right-hand side in (2.13) tend to 0 as $m, n \rightarrow \infty$. We have to show that the third term on the right also tends to 0 as $m, n \rightarrow \infty$.

To this effect, let $0 < h < \pi$ and $0 < v \leq \delta < \pi$ be given. By (2.4) and (2.5), in case $h < \delta$ we may write that

$$\begin{aligned} \tilde{f}^{10}(h; x, y+v) - \tilde{f}^{10}(h; x, y) &= \frac{1}{\pi} \int_h^\delta \frac{f(x-u, y+v) - f(x+u, y+v)}{2 \tan u/2} du \\ &\quad - \frac{1}{\pi} \int_h^\delta \frac{f(x-u, y) - f(x+u, y)}{2 \tan u/2} du \\ &\quad + \frac{1}{\pi} \int_\delta^\pi \frac{f(x-u, y+v) - f(x-u, y)}{2 \tan u/2} du \\ &\quad - \frac{1}{\pi} \int_\delta^\pi \frac{f(x+u, y+v) - f(x+u, y)}{2 \tan u/2} du. \end{aligned}$$

In case $h \geq \delta$, the first two terms on the right are missing, while \int_h^π is substituted for \int_δ^π in the last two terms. In any case, hence it follows that

$$\omega_y(\tilde{f}^{10}(h); \delta) \leq \frac{2}{\pi} \int_0^\delta \frac{\omega_x(f; 2u)}{u} du + \frac{2}{\pi} \int_\delta^\pi \frac{\omega_y(f; \delta)}{u} du,$$

and by (2.14),

$$\omega_y(\tilde{f}^{10}(h); \delta) = o \left\{ \log \frac{\pi}{\delta} \right\}^{-1} + \frac{2}{\pi} \omega_y(f; \delta) \log \frac{\pi}{\delta}. \quad (4.22)$$

Observe that the right-hand side of (4.22) is independent of h . The uniform convergence of $\tilde{s}_{mn}^{10}(f)$ now follows from (2.14), (2.15), (4.20)–(4.22). ■

Proof of Corollary 2.2.* In the special case of (2.16), the conjugate series (2.1) equals the product of the conjugate series to the Fourier series of g_1 (in x) and of the Fourier series of g_2 (in y). In particular,

$$\tilde{s}_{mn}^{10}(f; x, y) = \tilde{s}_m(g_1; x) s_n(g_2; y). \quad (4.23)$$

Likewise, we have

$$\begin{aligned} \tilde{f}^{10}(h; x, y) &= \tilde{g}_1(h; x) g_2(y), \quad h > 0, \\ \omega(f; \delta_1, \delta_2) &= \omega(g_1; \delta) \omega(g_2; \delta). \end{aligned}$$

Now, by the corresponding one-dimensional results of [7] and [2], we conclude the uniform convergence of $\tilde{s}_m(g_1; x)$ as $m \rightarrow \infty$ (due to (2.17)) and that of $s_n(g_2; y)$ as $n \rightarrow \infty$ (due to (2.18)). By (4.23), $\tilde{s}_{mn}^{10}(f; x, y)$ converges uniformly as $m, n \rightarrow \infty$. ■

Proof of Corollary 2.3. Clearly (2.20) follows from (2.12) in Theorem 2.1, by means of Lemmas 3.5 and 3.6. ■

Proof of Corollary 2.4.* First, we recall the convergence test given by Young [8]: If g is of bounded variation on \mathbb{T} , then $\tilde{s}_m(g; x)$ converges as $m \rightarrow \infty$ if and only if \tilde{g} exists at x . This will be applied in both parts below.

Sufficiency. Assume that $\tilde{f}^{10}(x, y) = \tilde{g}_1(x) g_2(y)$ exists. Since $g_2(y) \neq 0$, we conclude the existence of \tilde{g}_1 at x . By Young's test, $\tilde{s}_m(g_1; x)$ converges as $m \rightarrow \infty$. Being g_2 is of bounded variation, by the Dirichlet-Jordan test, $s_n(g_2; y)$ also converges as $n \rightarrow \infty$. Taking (4.23) into account, we are done.

Necessity. Assume that $\tilde{s}_{mn}^{10}(f; x, y)$ converges as $m, n \rightarrow \infty$. Since $s_n(g_2; y)$ converges to $g_2(y) \neq 0$ as $n \rightarrow \infty$, hence it follows that $\tilde{s}_m(g_1; x)$ also converges as $m \rightarrow \infty$. By Young's test, \tilde{g}_1 exists at x . Consequently, $\tilde{f}^{10}(x, y) = \tilde{g}_1(x) g_2(y)$ exists. ■

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